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Prepared by

Principle Investigator: Prof. Josef Shinar
Investigator: Dr. Isaac Forte

Faculty of Aeronautical Engineering
Technion - Israel Institute of Technology
Haifa, Israel

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Preface

This final scientific report summarizes a two-years investigation effort (1.9.1986-31.8.1988) sponsored by AFOSR Grant No. 86-0355. In order to enhance the usefulness of the report for individuals of different interests and responsibilities, it is divided into three parts.

The first part (I) outlines in generic terms the research concept which has lead to the development of an innovative approach for missile guidance law synthesis; it summarizes the main results achieved by the two-year effort and indicates directions deserving further investigation.

The second part (II), which can be used as an independent scientific document, concentrates on the theoretical aspects of the problem formulation and outlines the mathematical framework for the mixed guidance law synthesis.

The third part (III) is fully application oriented. It describes in detail the model used for the investigation and the process of interactive guidance law synthesis. Explicit guidelines for a potential user are given. Several examples, demonstrating the performance improvement which can be achieved by using the proposed approach, are also included in this part.

2.1 Introduction

In this part of the report, the terminal phase of a missile versus aircraft engagement in a noise corrupted environment is formulated as an imperfect information differential game. The pay-off of the game is the single shot kill probability of the missile, to be maximized by its designer and to be minimized by the pilot of the target aircraft. The optimal strategies in such a scenario can be mixed. This part of the report provides a vigorous mathematical framework for the analysis and outlines a constructive methodology for guidance law synthesis based on the concept of mixed strategies. Examples which demonstrate the applicability of the approach and demonstrate the improved performance are presented in part 3 of the report.

2.2 Problem's Formulation

2.2.1 Terminology

The terminology adopted in this work is as follows:

A pure strategy is a function which maps the information space into the control space. The pure strategy set of a player is the set which is formed by all the pure strategies of that player. A mixed strategy is a probability distribution on a pure strategy set.

A "guidance law" will be understood to be a function which maps the estimated state into acceleration commands, regardless of the form of the estimator used in the guidance loop. The combination of a guidance law and an estimator will be referred to as a "guidance policy" or a "pure guidance strategy". A "guidance

strategy" is a general name for a strategy which can be either pure or mixed.

2.2.2 Formulation of the Generalized Problem

In this section, the terminal phase of a future missile-aircraft encounter is formulated as a two-person, zero-sum, imperfect-information, differential game in which the allowable strategies of both players are mixed. The players are 1) the missile or the agent that fires it and 2) the pilot of the evading aircraft. For the sake of simplicity, we shall refer to them in the sequel as "the pursuer" and "the evader", respectively, and it will be understood that "the pursuer" also stands for the agent that fires the missile and that "the evader" includes the aircraft and its pilot, who makes the decisions.

In the formulation of the problem, the ECM capability of the evader, which may be necessary to enhance aircraft survivability, is taken into account in the form of "electronic jinking"[1]. This is a method which electronically generates an apparent motion of the aircraft's radar reflection center.

The state equation of the game is given by the nonlinear vector equation

$$\dot{x} = f(x,u,v) \quad ; \quad x(t_o) = x_o \tag{1}$$

where $x \in R^n$ is the state vector, $u \in U$ and $v \in V$ are the control vectors of the pursuer and evader respectively. u represents the commanded normal acceleration vector of the pursuer and v represents the normal acceleration vector of the evader. The game starts at t_o from initial conditions x_o and terminates at an

unspecified time t_f which satisfies

$$\dot{R}(t_f) = 0 \tag{2}$$

where R is the range between the pursuer and the evader.

The information structure of the game is given by:

$$z_e = h_e(x,\eta) \tag{3}$$

$$z_p = h_p(x,w,\xi) \tag{4}$$

where z_e and z_p are the measurement vectors of the evader and pursuer respectively. $w \in W$ is an additional control vector at the disposal of the evader. It represents an intentionally introduced disturbance, i.e. electronic jinking. η and ξ represent the respective measurement noise vectors.

The admissible control sets U, V and W are defined by:

$$U = \left\{u(t) \quad ; \quad \|u(t)\| \le (a_p)_{MAX} \quad \forall\, t \in [0,t_f]\right\} \tag{5}$$

$$V = \left\{v(t) \quad ; \quad \|v(t)\| \le (a_e)_{MAX}, \|\dot{v}(t)\| \le \alpha_L \quad \forall t \in [0,t_f]\right\} \tag{6}$$

$$W = \left\{w(t) \quad ; \quad \|w(t)\| \le w_{MAX} \quad \forall\, t \in [0,t_f]\right\} \tag{7}$$

where $(a_p)_{MAX}$ and $(a_e)_{MAX}$ are the pursuer's and evader's maximum lateral accelerations, respectively, w_{MAX} is the maximum possible disturbance due to the electronic jinking, and α_L is a parameter by which the roll dynamics of the evading aircraft are indirectly accounted for.

Let y be a random variable which takes the value of 1 in case the pursuer scores a kill and a value of zero in case it scores a

no-kill, then, the payoff function J is defined by:

$$J = E\{y\} \stackrel{\Delta}{=} 1 \cdot P(y=1) + 0 \cdot P(y=0) = P(y=1) = P_k \tag{8}$$

where P(y=1) is the probability that y=1. In other words, J is the single shot kill probability denoted by P_k.

Since the kill probability is a function of the miss distance, $R(t_f)$, J can be presented by:

$$J=E\{E\{y|R(t_f)\}\} = E\{P[y=1|R(t_f)]\} \stackrel{\Delta}{=} E\{P_k[R(t_f)]\} \tag{9}$$

where $P_k(\cdot)$ is a real valued function which describes the warhead lethality and which is subject to $0 \le P_k(\cdot) \le 1$.

The pursuer wishes to maximize J while the evader wishes to minimize it. This payoff function, which is indeed the one of true practical interest, has not been used in previous works.

The pure-strategy set of the pursuer, Δ_p is defined as a countable set of "guidance policies" of a predetermined structure. The different guidance policies in Δ_p result from different assumptions made on the target maneuver model. In general, each assumption leads to a different guidance law and estimator. More explicitly, assuming m_p different target maneuver models, Δ_p is given by

$$\Delta_p = \{\delta_{pj}, j=1,2\ldots,m_p\} \ , \ m_p \le \infty \tag{10}$$

where each pure strategy δ_{pj} is of the form

$$\delta_{pj} = (a_p)_{MAX} \, \mathrm{sat}\left[\frac{g_j(\hat{x}'^{(j)})}{(a_p)_{MAX}}\right] \in U \tag{11}$$

where g_j and $\hat{x}'^{(j)}$ are the j^{th} guidance law and output of the j^{th}

estimator, respectively.

Equation (11) simply states that δ_{pj} is a mapping, subject to some constraints, from the estimated state space to the control space. It is a generalized form for the guidance policies considered, also covering proportional navigation and other guidance policies discussed in previous works[2-6].

As far as the general formulation of the problem is concerned, there is no need at this point to specify any further the detailed structure of the guidance laws and estimators. It should be pointed out, however, that in any practical attempt to actually solve the problem, these structures will have to be determined in a fairly specific and detailed form.

Note that x' is the state vector required by the pure guidance strategy. x' is generally not identical to x. Nevertheless, some of its elements are also elements of x.

The pure-strategy set of the evader, Δ_e, is defined as a countable set of "actions" δ_{ei}, each of which is composed of a maneuver sequence and an electronic counter-measures policy. In other words,

$$\Delta_e = \{\delta_{ei}, i=1,2,\ldots,m_e\} \ , \ m_e \le \infty \tag{12}$$

where each pure strategy δ_{ei} is defined by the pair

$$\delta_{ei} = [v_i(t), w_i(t)] \qquad v_i(t) \in V \ , \ w_i(t) \in W \tag{13}$$

The game is played as follows: at the beginning of the game, or shortly prior to it, each player "chooses", through a chance mechanism, one of its pure strategies and plays accordingly until the end of the game. The chance mechanism, which determines the

pure strategy to be played, is a mechanization of the player's mixed strategy.

The selection by each player of one pure strategy at the outset actually transforms the problem into a matrix game.

In a matrix game the evader's and pursuer's mixed strategies are determined by sequences of real numbers, $\{\alpha_i\}_{i=1,\ldots m_e}$ and $\{\beta_j\}_{j=1,\ldots,m_p}$, respectively, which satisfy

$$\sum_{i=1}^{m_e} \alpha_i = 1, \quad \alpha_i \geq 0 \quad \forall\, i = 1,2,\ldots,m_e$$
$$\sum_{j=1}^{m_p} \beta_j = 1, \quad \beta_j \geq 0 \quad \forall\, j = 1,2,\ldots,m_p \tag{14}$$

where α_i determines the probability of "choosing" δ_{ei} by the evader and β_j determines the probability of "choosing" δ_{pj} by the pursuer. The payoff function, Eq. (9), in terms of Δ_e, Δ_p, $\{\alpha_i\}$, and $\{\beta_j\}$ is given by

$$J = J(\Delta_e, \{\alpha_i\}, \Delta_p, \{\beta_j\}) = \sum_{i=1}^{m_e} \sum_{j=1}^{m_p} \alpha_i, \beta_j, P_{ij}$$
$$\triangleq J_{\Delta_e, \Delta_p}(\{\alpha_i\}, \{\beta_j\}) \tag{15}$$

where P_{ij} is the SSKP for the case in which the pure strategies δ_{ei} and δ_{pj} are played, and it can be expressed by

$$P_{ij} = E\,\{P_k[R(t_f)]\,|\,\delta_{ei}, \delta_{pj}\} \tag{16}$$

Given the pure-strategy sets Δ_e and Δ_p, the solution of the game is presented by a triplet: the optimal sequences $\{\alpha_i^*\}\{\beta_j^*\}$,

called the optimal mixed strategies of the evader and the pursuer respectively, and a real number $0 \le V_m \le 1$, which is called the value of the game. The definition of V_m is

$$V_m = J_{\Delta_e,\Delta_p}(\{\alpha_i^*\},\{\beta_j^*\}) \tag{17}$$

The value satisfies a saddle-point inequality

$$J_{\Delta_e,\Delta_p}(\{\alpha_i^*\},\{\beta_j\}) \le V_m \le J_{\Delta_e,\Delta_p}(\{\alpha_i\},\{\beta_j^*\}) \tag{18}$$

for every arbitrary sequence $\{\alpha_i\}$ or $\{\beta_j\}$ satisfying Eq. (14).

Obviously, $\{\alpha_i^*\}$, $\{\beta_j^*\}$, and V_m are functions of Δ_e and Δ_p. Thus,

$$\begin{gathered} \{\alpha_i^*\} = \{\alpha_i^*(\Delta_e,\Delta_p)\} \ , \ \{\beta_j^*\} = \{\beta_j^*(\Delta_e,\Delta_p)\} \\ V_m = V_m(\Delta_e,\Delta_p) \end{gathered} \tag{19}$$

Motivated by Eq. (19), the generalized problem is formulated as follows:

Given an imperfect information pursuit-evasion game in which both the pursuer and the evader "select" at the outset a strategy from pure-strategy sets Δ_e and Δ_p of the form of Eqs. (10),(11) and (12),(13), respectively, and in which the payoff function is the single-shot kill probability given by Eq. (9), find the optimal pure strategy sets Δ_e^* and Δ_p^* that satisfy the following saddle-point relationship:

$$V_m(\Delta_e^*,\Delta_p) \le V_m(\Delta_e^*,\Delta_p^*) \le V_m(\Delta_e,\Delta_p^*) \tag{20}$$

for every admissible Δ_e and Δ_p.

2.2.3 Formulation of the Guidance Synthesis Problem

The mixed guidance strategy synthesis problem, which is the subject of the present study, is formulated as follows. For a given, but otherwise arbitrary pure strategy set of the evader, Δ_e, find the optimal pure strategy set of the pursuer $\tilde{\Delta}_p$, which satisfies

$$V_m(\Delta_e,\tilde{\Delta}_p) \geq V_m(\Delta_e,\Delta_p) \tag{21}$$

for every admissible Δ_p.

2.3 Mathematical Framework

2.3.1 New Notions

In this section the mathematical basis for the solution of the guidance strategy synthesis problem is layed down. First it is necessary to introduce several additional definitions that will be of use in the sequel.

A game in which the pure strategy sets of the pursuer and the evader are Δ_p and Δ_e respectively, will be referred to as a "pair (Δ_e,Δ_p)".

The entries of the payoff matrices and the corresponding optimal mixed strategies for the pairs (Δ_e,Δ_p) and $(\Delta_e,\tilde{\Delta}_p)$ will be denoted by $P_{ij},\{\alpha_i^*\},\{\beta_j^*\}$ and $\tilde{P}_{ij},\{\tilde{\alpha}_i\},\{\tilde{\beta}_j\}$ respectively.

A guidance policy δ_{P_j} will be said to be "active" in the pair (Δ_e,Δ_p) iff $\beta_j^*\neq 0$.

$\Delta_p^{(k)}$ will denote a pure strategy set of the pursuer which is composed of k elements.

The set of all sets $\Delta_p^{(k)}$ will be denoted by $D^{(k)}$, thus $D^{(k)}=\{\Delta_p^{(k)}\}$.

The set $\mathfrak{D}^{(k)}$ is defined by $\mathfrak{D}^{(k)} = \bigcup_{j=1}^{k} D^{(j)}$.

The set $\tilde{\Delta}_p^{(k)}$ which is composed of k active elements and which satisfies the inequality

$$V_m(\Delta_e,\tilde{\Delta}_p^{(k)}) \geq V_m(\Delta_e,\Delta_p) \;\forall\; \Delta_p \in \mathfrak{D}^{(k)} \tag{22}$$

will be called a "k optimal" set.

E and $E^{(k)}$ are sets of real numbers defined by:

$$E^{(k)} = \{x;x=V_m(\Delta_e,\Delta_p^{(k)}) \;\forall\; \Delta_p^{(k)} \in D^{(k)}\} \tag{23}$$

$$E = \{x;x=V_m(\Delta_e,\Delta_p) \;\forall\; \Delta_p \in \mathfrak{D}^{(\infty)}\} \tag{24}$$

If $\tilde{\Delta}_p^{(k)}$ and $\tilde{\Delta}_p$ exist, it is clear that

$$V_m(\Delta_e,\tilde{\Delta}_p^{(k)}) = \text{lub } E^{(k)} \tag{25}$$

$$V_m(\Delta_e,\tilde{\Delta}_p) = \text{lub } E \tag{26}$$

(lub - the least upper bound).

The optimal mixed strategies and the entries of the payoff matrix for the pair $(\Delta_e,\tilde{\Delta}_p^{(k)})$ will be denoted by $\{\tilde{\alpha}_i^{(k)}\}$, $\{\tilde{\beta}_j^{(k)}\}$ and $\tilde{P}_{ij}^{(k)}$ respectively.

2.3.2 <u>General Solution</u>

Throughout this section, it is assumed that $\tilde{\Delta}_p$ as well as $\tilde{\Delta}_p^{(k)}$ exist.

By definition, the value of the pair (Δ_e,Δ_p) is given by:

$$V_m(\Delta_e,\Delta_p) = \sum_{i=1}^{m_e} \alpha_i^* \sum_{j=1}^{m_p} \beta_j^* P_{ij} = \min_{1\leq i\leq m_e} \sum_{j=1}^{m_p} \beta_j^* P_{ij} \tag{27}$$

Since m_p is determined for each Δ_p and since $\{\alpha_i^*\}, \{\beta_j^*\}$ and P_{ij} are all functions of Δ_p for a given Δ_e, Eq. (27) can be written as:

$$V_m(\Delta_e, \Delta_p) = \min_{1 \le i \le m_e} \sum_{j=1}^{m_p(\Delta_p)} \beta_j^*(\Delta_p) P_{ij}(\Delta_p) \tag{28}$$

By rewriting Eq. (21) as:

$$V_m(\Delta_e, \tilde{\Delta}_p) = \max_{\Delta_p \in \mathfrak{D}^{(\infty)}} \{V_m(\Delta_e, \Delta_p)\} \tag{29}$$

and by substituting Eq. (28) into Eq. (29) we obtain

$$V_m(\Delta_e, \tilde{\Delta}_p) = \max_{\Delta_p \in \mathfrak{D}^{(\infty)}} \min_{1 \le i \le m_e} \sum_{j=1}^{m_p(\Delta_p)} \beta_j^*(\Delta_p) P_{ij}(\Delta_p) \tag{30}$$

Thus,

$$\tilde{\Delta}_p = \arg \max_{\Delta_p \in \mathfrak{D}^{(\infty)}} \left\{ \min_{1 \le i \le m_e} \sum_{j=1}^{m_p(\Delta_p)} \beta_j^*(\Delta_p) P_{ij}(\Delta_p) \right\} \tag{31}$$

Equation (31) is the formal solution of the mixed guidance law synthesis problem (Eq. (21)), and it states that $\tilde{\Delta}_p$ is the pure strategy set which maximizes the guaranteed single shot kill probability.

Since $\beta_j^*(\cdot)$ and $P_{ij}(\cdot)$ are unkown implicit functions of Δ_p, and since the number of elements in $\tilde{\Delta}_p$ is not known in advance, it is clear that $\tilde{\Delta}_p$ cannot be computed directly from Eq. (31). Thus, some kind of iterative algorithm is necessary to obtain the solution. In this paper an iterative algorithm, based on the sequential computation of $\tilde{\Delta}_p^{(k)}$ with k progressively increasing from 1 (until the optimum is reached), is proposed.

2.3.3 Properties of $\tilde{\Delta}_p^{(k)}$

In this subsection it is proved that the set $\tilde{\Delta}_p^{(k)}$, if it exists, must have the following properties:

Property P1: The requirement that $\tilde{\Delta}_p^{(k)}$ be composed of k active elements in the pair $(\Delta_e, \tilde{\Delta}_p^{(k)})$ leads to:

$$\sum_{i=1}^{m_e} \tilde{\alpha}_i^{(k)} \tilde{P}_{ij}^{(k)} = V_m(\Delta_e, \tilde{\Delta}_p^{(k)}) \quad , \forall \; 1 \le j \le k \tag{P1}$$

Property P2: $\tilde{\Delta}_p^{(k)}$ must maximize, within the limitations imposed by $\mathfrak{D}^{(k)}$, the guaranteed single shot kill probability, namely:

$$\tilde{\Delta}_p^{(k)} = \arg \max_{\Delta_p \in D^{(k)}} \left\{ \min_{i \le i \le m_e} \sum_{j=1}^{k} \beta_j^* \, (\Delta_p^{(k)}) P_{ij} (\Delta_p^{(k)}) \right\} \tag{P2}$$

Proof of P1: Define C_j by:

$$C_j = \sum_{i=1}^{m_e} \tilde{\alpha}_i^{(k)} \tilde{P}_{ij}^{(k)} \tag{32}$$

By substituting (32) into the left-hand side of the saddled point inequality (18) one obtains:

$$V_m(\Delta_e, \tilde{\Delta}_p^{(k)}) = \sum_{j=1}^{k} \tilde{\beta}_j^{(k)} C_j \ge \sum_{j=1}^{k} \beta_j C_j \quad \forall \quad \{\beta_j\} \tag{33}$$

If all the C_j's are not equal, there must be a j^* such that

$$C_{j^*} = \max_{i \le j \le k} C_j \tag{34}$$

Thus, since $\sum_{j=1}^{k} \tilde{\beta}_j^{(k)} = 1$ and since $\tilde{\beta}_j^{(k)} \ne 0 \;\forall\; j$ (which is necessary for k active elements), it is clear that

$$C_{j^*} > \sum_{j=1}^{k} \tilde{\beta}_j^{(k)} C_j \tag{35}$$

But, by selecting $\{\hat{\beta}_j\}$ such that $\hat{\beta}_{j^*}=1$ (and $\hat{\beta}_j=0 \;\forall\; j \neq j^*$) and by substituting it into (33) we obtain

$$C_{j^*} \leq \sum_{j=1}^{k} \tilde{\beta}_j^{(k)} C_j \tag{36}$$

which is a contradiction to (35). This leads to the conclusion that all the C_j's must be equal, say $C_j = C \;\forall\; j$. Thus, Eqs. (32) and (33) lead to

$$V_m(\Delta_e, \tilde{\Delta}_p^{(k)}) = \sum_{j=1}^{k} \tilde{\beta}_j^{(k)} C = C = \sum_{i=1}^{m_e} \tilde{\alpha}_i^{(k)} \tilde{P}_{ij}^{(k)} \;\forall\; i \leq j \leq k \tag{37}$$

which concludes the proof.

Proof of P2: By using the same steps which led to the definition of $\tilde{\Delta}_p$ in Eq. (31) as the solution of the problem posed in (21), it can be shown that $\tilde{\Delta}_p^{(k)}$ which satisfied (22) is given by:

$$\tilde{\Delta}_p^{(k)} = \arg \max_{\Delta_p \in D^{(k)}} \left\{ \min_{i \leq i \leq m_e} \sum_{j=1}^{m_p(\Delta_p)} \beta_j^* (\Delta_p) P_{ij}(\Delta_p) \right\} \tag{38}$$

But, since by definition $\tilde{\Delta}_p^{(k)} \in D^{(k)}$ it is clear that

$$\tilde{\Delta}_p^{(k)} = \arg \max_{\Delta_p \in D^{(k)}} \left\{ \min_{i \leq i \leq m_e} \sum_{j=1}^{k} \beta_j^* (\Delta_p^{(k)}) P_{ij}(\Delta_p^{(k)}) \right\} \tag{39}$$

which concludes the proof.

2.3.4 Derivation of the Algorithm for Computing $\tilde{\Delta}_p^{(M)}$ (or $\tilde{\Delta}_p$)

This derivation is based on the following theorems:

Theorem 1: For every $\varepsilon > 0$, there exists an integer $M(\varepsilon)$ and a finite set $\tilde{\Delta}_p^{(M)}$ such that $V_m(\Delta_e, \tilde{\Delta}_p) - V_m(\Delta_e, \tilde{\Delta}_p^{(M)}) \le \varepsilon$.

Proof: Let us take a sequence $\{t_i\} \in E$ such that $t_i = V_m(\Delta_e, \tilde{\Delta}_p^{(i)})$. By virtue of (22) $\{t_i\}$ is monotonically increasing, and since $V_m(\cdot, \cdot)$ is bounded so is $\{t_i\}$. Therefore $\{t_i\}$ converges in R^1. Let's say $t_i \to t$. Let us take an arbitrary series $\{a_i\} \in E$ such that $a_i = V_m(\Delta_e, \Delta_p^{(i)})$. From (22) it is clear that $a_i \le t_i$ for all i. thus

$$a_j < t \quad \forall \quad a_i \in E \quad \forall \quad i \tag{40}$$

therefore t=lub E. By making use of (26) we obtain

$$t = \text{lub } E = V_m(\Delta_e, \tilde{\Delta}_p) \tag{41}$$

This means that $V_m(\Delta_e, \tilde{\Delta}_p^{(i)}) \to V_m(\Delta_e, \tilde{\Delta}_p)$. Therefore, for every $\varepsilon > 0$ there exists an integer $M = M(\varepsilon)$ such that

$$V_m(\Delta_e, \tilde{\Delta}_p) - V_m(\Delta_e, \tilde{\Delta}_p^{(M)}) \le \varepsilon \tag{42}$$

This concludes the proof of the theorem.

Theorem 2: If $V_m(\Delta_e, \tilde{\Delta}_p^{(k+1)}) > V_m(\Delta_e, \tilde{\Delta}_p^{(k)})$, then for every $\{\tilde{\delta}_i^{(k)}\}$ there exists a strategy $\delta_{p\ell} \notin \tilde{\Delta}_p^{(k)}$ such that

$$\sum_{i=1}^{m_e} \tilde{\delta}_i^{(k)} P_{i\ell} > V_m(\Delta_e, \tilde{\Delta}_p^{(k)}).$$

Proof: Assume that

$$V_m(\Delta_e, \tilde{\Delta}_p^{(k+1)}) > V_m(\Delta_e, \tilde{\Delta}_p^{(k)}) \tag{43}$$

and that the theorem is not true. This means that there exists a mixed strategy $\{\alpha_i^{(k)}\}$ such that

$$\sum_{i=1}^{m_e} \tilde{\alpha}_i^{(k)} P_{i\ell} \le V_m(\Delta_e, \tilde{\Delta}_p^{(k)}) \quad \forall \quad \delta_{p\ell} \in \tilde{\Delta}_p^{(k)} \tag{44}$$

But because of the optimality of $\{\tilde{\alpha}_i^{(k)}\}$ it is also true that

$$\sum_{i=1}^{m_e} \tilde{\alpha}_i^{(k)} P_{ij} \le V_m(\Delta_e, \tilde{\Delta}_p^{(k)}) \quad \forall \quad \delta_{p_j} \in \tilde{\Delta}_p^{(k)} \tag{45}$$

thus by combining (44) and (45) we conclude that

$$\sum_{i=1}^{m_e} \tilde{\alpha}_i^{(k)} P_{ij} \le V_m(\Delta_e, \tilde{\Delta}_p^{(k)}) \tag{46}$$

for all admissible δ_{p_j}. Since $V_m(\Delta_e, \tilde{\Delta}_p^{(k+1)})$ is the value of the pair $(\Delta_e, \tilde{\Delta}_p^{(k+1)})$ it satisfies the following (see, (15),(17),(18)):

$$\begin{aligned} V_m(\Delta_e, \tilde{\Delta}_p^{(k+1)}) &= \sum_{i=1}^{m_e} \sum_{j=1}^{k+1} \tilde{\alpha}_i^{(k+1)} \tilde{\beta}_j^{(k+1)} \tilde{P}_{ij}^{(k+1)} \le \\ &\le \sum_{i=1}^{m_e} \sum_{j=1}^{k+1} \alpha_i \tilde{\beta}_j^{(k+1)} \tilde{P}_{ij}^{(k+1)} = \\ &= \sum_{j=1}^{k+1} \tilde{\beta}_j^{(k+1)} \sum_{i=1}^{m_e} \alpha_i \tilde{P}_{ij}^{(k+1)} \end{aligned} \tag{47}$$

for every admissible $\{\alpha_i\}$, and in particular for $\{\tilde{\alpha}_i^{(k)}\}$. By combining (46) and (47) we get:

$$V_m(\Delta_e,\tilde{\Delta}_p^{(k+1)}) \le \sum_{j=1}^{k+1} \tilde{\beta}_j^{(k+1)} \sum_{i=1}^{m_e} \tilde{\alpha}_i^{(k)} \tilde{P}_{ij}^{(k+1)} \le$$

$$\le \sum_{j=1}^{k+1} \tilde{\beta}_j^{(k+1)} V_m(\Delta_e,\tilde{\Delta}_p^{(k)}) = \qquad (48)$$

$$= V_m(\Delta_e,\tilde{\Delta}_p^{(k)})$$

which contradicts (43). Thus the theorem is proved.

2.4 Iterative Design Algorithm

Based on the theorems presented previously, the following iterative algorithm is proposed for finding $\tilde{\Delta}_p$.

Step 1 - Set k=1.

Step 2 - Find $\tilde{\Delta}_p^{(k)}$ which satisfies P1 and P2.

Step 3 - Solve the pair $(\Delta_e,\tilde{\Delta}_p^{(k)})$ yielding $\{\tilde{\alpha}_i^{(k)}\}$, $\{\tilde{\beta}_j^{(k)}\}$ and $V_m(\Delta_e,\tilde{\Delta}_p^{(k)})$. $\{\tilde{\alpha}_i^{(k)}\}$, $\{\tilde{\beta}_j^{(k)}\}$ may not be unique.

Step 4 - For every $\{\tilde{\alpha}_i^{(k)}\}$ search for a strategy $\delta_{p_\ell} \notin \tilde{\Delta}_p^{(k)}$ such that $\sum_{i=1}^{m_e} \tilde{\alpha}_i^{(k)} P_{i\ell} > V_m(\Delta_e,\tilde{\Delta}_p^{(k)})$. If no such δ_{p_ℓ} is found for <u>every</u> $\{\tilde{\alpha}_i^{(k)}\}$ then $\tilde{\Delta}_p=\tilde{\Delta}_p^{(k)}$ and go to Step 6, otherwise go to Step 5.

Step 5 - If k=1 set k=k+1 and go to 2, otherwise check if $V_m(\Delta_e,\tilde{\Delta}_p^{(k)}) - V_m(\Delta_e,\tilde{\Delta}_p^{(k-1)}) > \delta$ for a given δ. If so, set k=k+1 and go to 2, otherwise set M=k and go to 6.

Step 6 - Stop.

2.5 Concluding Remarks

This part outlines the mathematical framework for the synthesis of optimal mixed guidance strategies. The properties of the "k-optimal" strategy sets are derived and proven. It is also demonstrated that these sets converge to the optimal pure strategy set. This implies that even an infinite optimal strategy set can be approximated by a finite "k-optimal" set. Moreover, a constructive iterative procedure for finding the finite approximation of the optimal strategy set is presented. This procedure provides guidelines for the designer to develop improved missile guidance laws based on the mixed strategy concept. This procedure has been implemented in a set of examples for a first-order dynamics skid-to-turn missile in Part II of this report.

2.6 References

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